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A Comparison of Numerical Solutions to the Inviscid Equations of Fluid Motion

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The second-order MacCormack method and the third-order Rusanov and Kutler–Warming–Lomax methods are applied to the inviscid 1-d Burgers' equation, wedge flow and the problem of shock reflection from a rigid boundary. The numerical solution in each case is compared to the exact solution and the quantitative estimates of accuracy are obtained. Results of this study show that the third-order Kutler–Warming–Lomax or the Rusanov methods tuned for minimum dissipation or minimum dispersion provide the most accurate solution in each of the examples considered.

INTRODUCTION

The direct integration of the equations of motion governing fluid flow has become common place since the advent of the high-speed digital computer. The number of investigators in the field has grown and the number of finite difference methods available has increased until the selection of a numerical method for a given problem has become a difficult part of the problem's solution.

This study compares the second-order MacCormack [11], third-order Rusanov [13], and the third-order Kutler–Warming–Lomax [6] methods on the basis of execution time, resolution, and ease of coding. All of the methods tested are explicit and all applications were to hyperbolic partial differential equations, that is, unsteady methods used to determine steady-state flows or spatially steady supersonic flows whose downstream solutions can be treated as initial value problems.

The examples treated in this paper all involve propagation of discontinuities either in two-dimensional space or time and one space dimension. The examples chosen include the propagation of a double discontinuity in 1-d space using Burgers' equation [5], the solution of the equations of simple wedge flow using a radially asymptotic approach, and the reflection of a shockwave from a solid boundary in a supersonic stream. The exact details of each of these problems is discussed in later sections.

Emery [3] has evaluated several first- and second-order methods including the Lax [8], Rusanov first-order [1,4], Lanshoff [10], Lax–Wendroff [9] and Richtmeyer [12] techniques. He concluded that, of the methods tested, the first-order Rusanov scheme provided the most satisfactory results. Emery's study included propagation of a one-dimensional shockwave through a perfect gas and its reflection from a wall, flow over a two-dimensional flat faced step and a circular cone in a high Mach number flow. Taylor, Ndefo, and Masson [15] have recently written a paper in which they have evaluated the first-order methods of Godunov [4] and Rusanov, the second-order methods of MacCormack and Richtmeyer, and the third-order method of Rusanov. Their investigation included results from both inviscid and viscous examples. The inviscid examples, however, were confined to a propagating shock wave in one dimension, a rarefaction wave and a contact discontinuity. Taylor, Ndefo and Masson concluded that of the methods examined, the third-order Rusanov method provided the most accurate results although it does require at least twice as much computer time as the first-order methods.

Kutler, Warming, and Lomax recently introduced a noncentered third-order method which they used in calculating space shuttle flow fields. In addition they attempted to "tune" the method to minimize either dispersion or dissipation produced in the solution. This was achieved by varying the required stability parameter of third-order methods to cause the phase shift or damping to be a minimum at each point in the computation. The results of their study appear to show that the minimum dispersion case provides the best solution.

In the following sections, a brief review of each differencing method is presented, application to typical problems is made, and finally, results and conclusions based on this study are discussed.

Numerical Methods

The numerical techniques used in this study have been developed for hyperbolic systems of the form:

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0, \tag{1}$$

where E and F may be n component vectors and x may be of arbitrary dimension. The techniques are based upon the Runge–Kutta method and are explicit in advancing the solution forward in time.

MacCormack's Method

MacCormack [11] developed a noncentered second-order version of the Lax–Wendroff method which has been used extensively in solving gas dynamic problems.

This technique consists of a two-step predictor–corrector sequence of the form:

$$\begin{aligned} \tilde{E}_j^{n+1} &= E_j^{\,n} - \frac{\Delta T}{\Delta X}[1 - \epsilon\nabla]\,\Delta F_j^{\,n}, \\ E_j^{n+1} &= \frac{1}{2}[E_j^{\,n} + \tilde{E}_j^{n+1}] - \frac{1}{2}\frac{\Delta T}{\Delta X}(1 + \epsilon\Delta)\,\nabla\tilde{F}_j^{n+1}, \end{aligned} \tag{2}$$

where the tilde represents values at the intermediate step and

$$\begin{aligned} \Delta F_j &= F_{j+1} - F_j\,, \\ \nabla F_j &= F_j - F_{j-1}\,. \end{aligned} \tag{3}$$

The value of ϵ can be either 0 or 1. If $\epsilon = 0$, the predictor uses a forward difference and the corrector uses a backward difference on F_j. If $\epsilon = 1$ the differencing in the predictor and corrector is reversed. Only the case where $\epsilon = 0$ was evaluated in this study.

The stability of this method was investigated in detail by MacCormack and his results show that stability is assured if

$$|\,\nu\,| \leqslant 1, \tag{4}$$

where ν is the Courant number given by

$$\nu = \sigma(\Delta T/\Delta X), \tag{5}$$

and σ represents the maximum eigenvalue of the Jacobian matrix $\partial F/\partial E$ which arises in the stability analysis of Eq. (1).

Rusanov's Method

Rusanov [13] and Burstein and Mirin [2] simultaneously developed a third-order scheme based upon application of the Runge–Kutta method. This method uses central differencing and when applied to Eq. (1) becomes

$$\begin{aligned} E_{j+1/2}^{(1)} &= \frac{1}{2}[E_{j+1}^{n} + E_j^{\,n}] - \frac{1}{3}\frac{\Delta T}{\Delta X}[F_{j+1}^{n} - F_j^{\,n}], \\ E_j^{(2)} &= E_j^{\,n} - \frac{2}{3}\frac{\Delta T}{\Delta X}[F_{j+1/2}^{(1)} - F_{j-1/2}^{(1)}], \\ E_j^{n+1} &= E_j^{\,n} - \frac{1}{24}\frac{\Delta T}{\Delta X}[-2F_{j+2}^{(n)} + 7F_{j+1}^{(n)} - 7F_{j-1}^{(n)} + 2F_{j-2}^{(n)}] \\ &\quad - \frac{3}{8}\frac{\Delta T}{\Delta X}[F_{j+1}^{(2)} - F_{j-1}^{(2)}] \\ &\quad - \frac{\omega}{24}[E_{j+2}^{n} - 4E_{j+1}^{n} + 6E_j^{\,n} - 4E_{j-1}^{n} + E_{j-2}^{n}]. \end{aligned} \tag{6}$$

The omega term in the third step is a fourth-order difference required for stability. Since it is fourth-order, the third-order accuracy of the method is unaffected by its addition.

The stability analysis of this technique has been carried out in detail by Burstein and Mirin. Their results show that stability of the system is assured if

$$|\nu| \leqslant 1$$

and

$$4\nu^2 - \nu^4 \leqslant \omega \leqslant 3. \tag{7}$$

The Kutler–Lomax–Warming (K–L–W) *Method*

Kutler, Lomax, and Warming [6] have developed a noncentered version of the Rusanov scheme. There are two major differences between this technique and the original third-order Rusanov method.

The K–L–W method uses noncentered differences and uses the MacCormack method (evaluated at 2/3 ΔT) for the first two steps while the Rusanov third level is used. The other difference is that the fourth-order ω term has been differenced in a conservative manner so that the value of ω can be altered during the calculations. If the ω term is differenced as in the Rusanov method and altered during computation, incorrect wave speeds are produced in the numerical solution.

The K–L–W method applied to Eq. (1) takes the form:

$$E_j^{(1)} = E_j^{\,n} - \frac{2}{3}\frac{\Delta T}{\Delta X}[(1-\epsilon)F_{j+1}^n - (1-2\epsilon)F_j^{\,n} - \epsilon F_{j-1}^n],$$

$$E_j^{(2)} = \frac{E_j^{\,n} + E_j^{(1)}}{2} - \frac{1}{3}\frac{\Delta T}{\Delta X}[\epsilon F_{j+1}^{(1)} - (1-2\epsilon)F_j^{(1)} + (\epsilon - 1)F_{j-1}^{(1)}],$$

$$\begin{aligned} E_j^{n+1} = E_j^{\,n} &- \frac{\omega_{j+1/2}^n}{24}[E_{j+2}^n - 3E_{j+1}^n + 3E_j^{\,n} - E_{j-1}^n] \\ &+ \frac{\omega_{j-1/2}^n}{24}[E_{j+1}^n - 3E_j^{\,n} + 3E_{j-1}^n - E_{j-2}^n] \\ &- \frac{1}{24}\frac{\Delta T}{\Delta X}[-2F_{j+2}^n + 7F_{j+1}^n - 7F_{j-1}^n + 2F_{j-2}^n] \\ &- \frac{3}{8}\frac{\Delta T}{\Delta X}[F_{j+1}^{(2)} - F_{j-1}^{(2)}], \end{aligned} \tag{8}$$

where functionally the ω values are represented by

$$\begin{aligned} \omega_{j+1/2}^n &= \omega(\nu_{j+1/2}^n), \\ \omega_{j-1/2}^n &= \omega(\nu_{j-1/2}^n), \end{aligned} \tag{9}$$

where

$$\nu_{j+1/2}^{n} = \frac{1}{4}\,[\sigma_{j+2}^{n} + \sigma_{j+1}^{n} + \sigma_{j}^{n} + \sigma_{j-1}^{n}]\,\frac{\Delta T}{\Delta X},$$
$$\nu_{j-1/2}^{n} = \frac{1}{4}\,[\sigma_{j+1}^{n} + \sigma_{j}^{n} + \sigma_{j-1}^{n} + \sigma_{j-2}^{n}]\,\frac{\Delta T}{\Delta X}. \tag{10}$$

This method has the same stability bounds as the Rusanov method.

Kutler, Lomax, and Warming tried to "tune" the two ω parameters in an attempt to minimize either the dispersion or dissipation present at each step in the calculation. The choice of either minimum dispersion or minimum dissipation determines the precise functional form as noted in Eq. (9). These functional forms are determined by an analysis of the linear first-order wave equation

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0. \tag{11}$$

Application of the K–L–W method to this equation results in the modified partial differential equation

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = -\frac{c}{24}\,\Delta X^{3}\left[\frac{\omega}{\nu} - 4\nu + \nu^{3}\right]\frac{\partial^{4} u}{\partial x^{4}}$$
$$- \frac{c}{120}\,\Delta X^{4}[-5\omega + (4\nu^{2} + 1)(4 - \nu^{2})]\,\frac{\partial^{5} u}{\partial x^{5}} + \cdots. \tag{12}$$

The term in the modified equation representing dispersion is a fifth derivative term while the fourth derivative represents dissipation. If one wishes to reduce the dissipation of the method, the ω parameter should be selected to minimize the fourth derivative term. The obvious choice is

$$\omega = 4\nu^{2} - \nu^{4}, \tag{13}$$

which is the lower stability bound as predicted by linear theory. If minimum dispersion is desired, the coefficient of the fifth derivative is set equal to zero giving

$$\omega = \frac{(4\nu^{2} + 1)(4 - \nu^{2})}{5}. \tag{14}$$

Results obtained by Kutler, Warming, and Lomax show that an improvement in the solution is obtained by altering ω according to either of these schemes. It should be noted that the value of ν appearing in Eq. (9) and used in Eq. (10) is the local value of the Courant number based on an average of the eigenvalues at the mesh points in use for that calculation. The ν values are obtained by using Eq. (10)

and the appropriate values of ω are then determined through Eq. (13) or (14). This method accounts for the changes in eigenvalue structure encountered during the computation procedure.

Solution of the Modified Burgers' Equation

The hyperbolic form of the equation introduced by Burgers [5] is a valuable aid for use in studying the behavior of a given numerical method when applied to a nonlinear equation. A particularly useful analog of the fluid flow equations is the modified or inviscid form of the Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2/2)}{\partial x} = 0. \tag{15}$$

Kutler [7] has successfully used this equation as an analog of the inviscid Euler equations and studied solutions produced using various numerical algorithms. In particular, Kutler examined first- and second-order methods used in solving gas dynamic problems. He concluded that MacCormack's method provided the most satisfactory results from among those he examined. Since that investigation, third-order methods have been developed and it is of importance to compare solutions of Burgers' equation obtained using third-order methods with those obtained using second-order methods.

The problem studied is to determine the solution of Eq. (15) subject to the initial conditions,

$$\begin{aligned} u &= 0 \qquad & x &\geqslant x_1, \\ u &= u_1 \qquad & x_1 &> x > x_2, \\ u &= u_2 \qquad & x &\leqslant x_2, \end{aligned} \tag{16}$$

where $u_2 > u_1$. Since this problem represents the intersection of two discontinuities, the exact solution must be represented in two regions: the first region is prior to the intersection of the discontinuities and the second is after the intersection. The exact solution in these regions is

Region 1

$$\begin{aligned} u(x, t) &= 0 \qquad & &(x - x_1)/t > u_1/2, \\ u(x, t) &= u_1 \qquad & &x_2 + ((u_1 + u_2)/2)\, t \leqslant x \leqslant x_1 + u_1 t/2, \\ u(x, t) &= u_2 \qquad & &(x - x_2)/t < (u_1 + u_2)/2; \end{aligned}$$

Region 2

$$\begin{aligned} u(x, t) &= 0 \qquad & &x/t > u_2/2, \\ u(x, t) &= u_2 \qquad & &x/t \leqslant u_2/2. \end{aligned} \tag{17}$$

The first double shock considered was with $u_1 = 2$ and $u_2 = 5$. The mesh used was 100 points in length in the x-direction while $x_1 = 36$ and $x_2 = 15$.

The results of applying MacCormack's and Rusanov's methods to this problem are shown in Fig. 1. The Courant number is unity in both cases while the stability

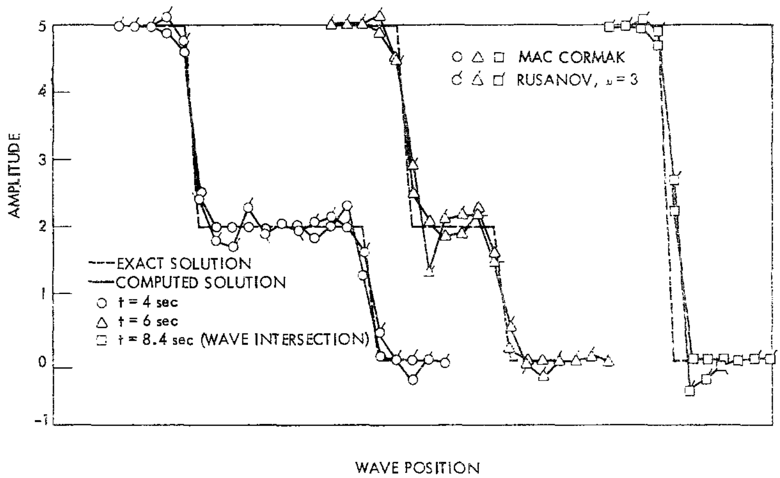

FIG. 1. Solution of Burgers' equation for 5-2 double shock $\Delta T/\Delta x = 0.2$, $v = 1.0$.

parameter in the Rusanov method is chosen to be 3 as required by the linear stability analysis. Both methods predict the wave position and amplitude correctly. The results show that better wave resolution is provided by MacCormack's method at least for the effective Courant numbers considered. It should be noted that the effective Courant number is 0.4 at the lower wave for this case. The wave front resolution at the time of intersection of the two waves is particularly sharp using MacCormack's method. There is no overshoot on the wave front, and the wave is spread over approximately two cells. This is probably due to the noncentered differencing in this method.

Figures 2 and 3 show results obtained for the same problem run with a Courant number of 0.5 and the ω parameter at its upper and lower bounds for Rusanov's method. For this particular problem the effective Courant number is 0.2 at the lower discontinuity. The effect of changing the mesh size becomes apparent in this case. Contrary to the previous conclusion, either of the Rusanov solutions is superior to that obtained using MacCormack's method. The third-order technique produces a better solution over the range of smaller Courant numbers encountered.

An objective comparison of results using MacCormack's, Rusanov's, and the K–L–W methods can only be made if a quantitative measure of the accuracy of the

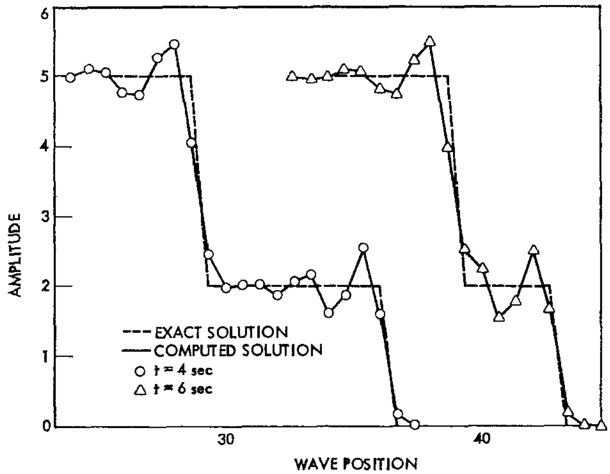

FIG. 2. Solution of Burgers' equation for 5-2 double shock MacCormack Method $\Delta T/\Delta x = 0.1$, $v = 0.5$.

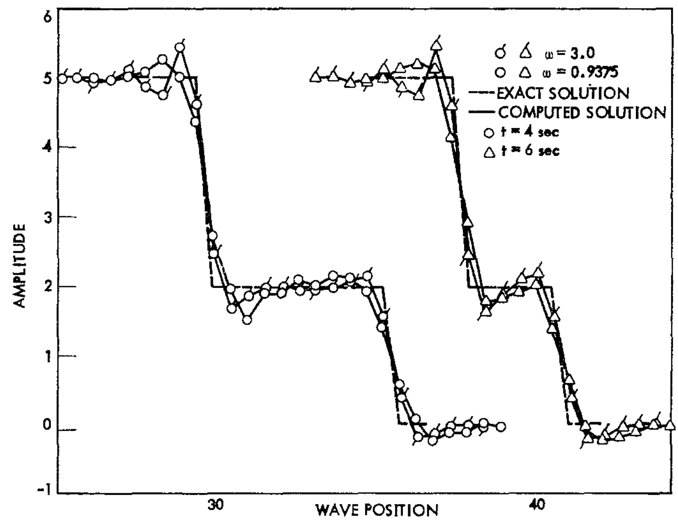

FIG. 3. Solution of Burgers' equation for 5-2 double shock Rusanov Method $\Delta T/\Delta x = 0.1$, $\nu = 0.5$.

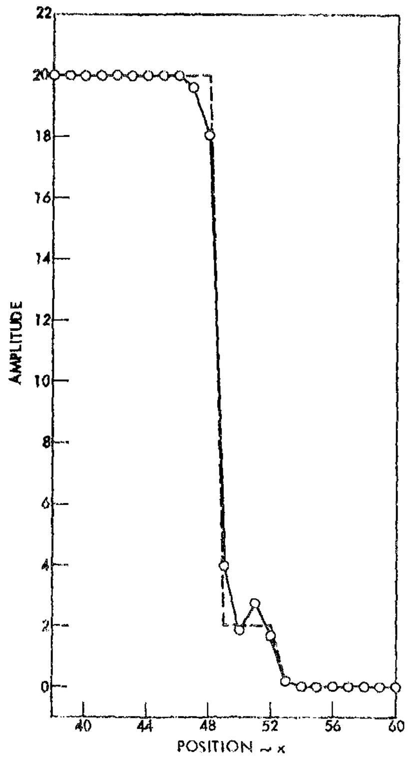

FIG. 4. Solution of Burgers' equation for 20-2 double shock MacCormack Method $\Delta T/\Delta x = 0.05$, $\nu = 1.0$.

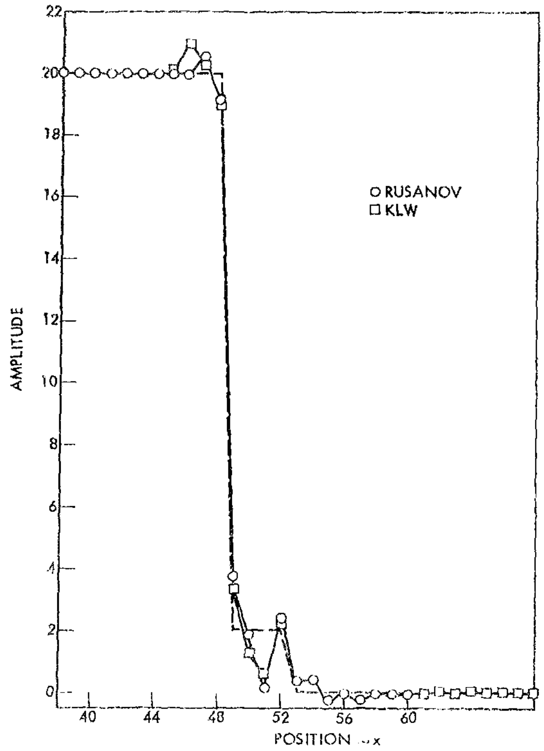

FIG. 5. Solution of Burgers' equation for 20-2 double shock with $\Delta T/\Delta x = 0.05$, $\nu = 1.0$, $\omega = 3.0$.

numerical solution can be obtained. Since the exact solution to Burgers' equation is known, a measure of the accuracy can be

$$\text{Error} = \int_t \int_x | u_{\text{computed}} - u_{\text{exact}} | \, dx \, dt. \tag{18}$$

Using this definition of error, a 20 — 2 discontinuity was used for initial conditions, and solutions of the Burgers' equation were obtained using the three methods being evaluated, including the minimum dispersion and dissipation cases of the K–L–W method.

The solutions obtained for fixed ω are shown in Figs. 4 and 5, and the minimum dispersion and dissipation cases are shown in Fig. 6. The results obtained using the tuned third-order methods are clearly better than those using constant ω. However, the MacCormack solution appears to be better than either of the Rusanov or the K–L–W cases. This is apparent not only by visual inspection of the

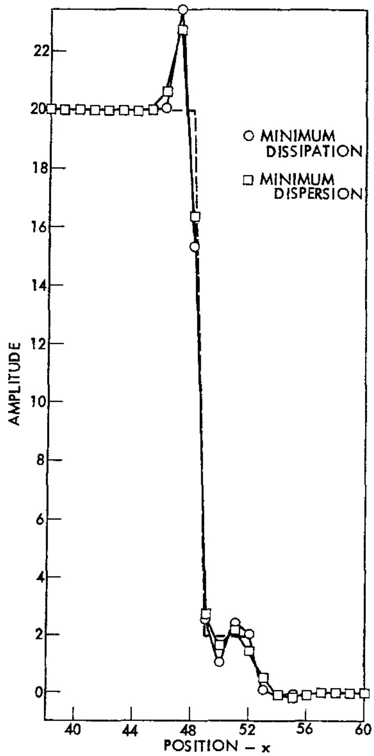

FIG. 6. Solution of Burgers' equation for 20-2 double shock, Tuned K–L–W method $\Delta T/\Delta x = 0.05$, $\nu = 1.0$.

results, but the error estimate based on Eq. (18) and tabulated in Table I also shows this to be the case. The difference between the minimum dispersive and minimum dissipative cases is not large and one gains little in selecting one over the other in this example.

TABLE I

Comparison of Results for Burgers' Equation

Differencing method			Relative computational time required	Estimate of error [Eq. (18)]
5-2 discontinuity problem				
MacCormack	$v = 1$		1	—
Rusanov	$v = 1$	$\omega = 3$	2.25	—
20-2 discontinuity problem				
MacCormack	$v = 1$		1	48.738
Rusanov	$v = 1$	$\omega = 3$	2.25	78.098
K–L–W	$v = 1$	$\omega = 3$	2.25	76.651
K–L–W	$v = 1$	ω variable (minimum dispersion)	2.758	57.773
K–L–W	$v = 1$	ω variable (minimum dissipation)	2.758	52.079

The Rusanov and K–L–W cases show nearly the same error with the K–L–W method being somewhat more accurate. In view of the ease of programming and the slight improvement in the solution, the K–L–W technique would be the better choice in a practical application.

Wedge Flow

Supersonic flow over a two-dimensional wedge provides a very simple example requiring the solution of the equations of motion. Again, valid comparisons of accuracy can be made since the exact solution for wedge flow is known.

The equations of motion governing inviscid flow of a perfect gas over a pointed two-dimensional wedge are

$$\begin{aligned}
\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} &= 0, \\
\frac{\partial (P + \rho u^2)}{\partial x} + \frac{\partial \rho u v}{\partial y} &= 0, \\
\frac{\partial (\rho u v)}{\partial x} + \frac{\partial (P + \rho v^2)}{\partial y} &= 0, \\
H_T = \frac{\gamma}{\gamma - 1} P/\rho &+ \frac{u^2 + v^2}{2}.
\end{aligned} \tag{19}$$

For this study, these equations were transformed into a polar coordinate system with polar radius measured from the wedge vertex and polar angle measured from the wedge centerline serving as independent variables. However, the rectangular velocity components as written in Eq. (19) were used rather than polar components. The polar coordinates (r, θ) form a more convenient set in which to apply the boundary conditions at the wedge surface. At the body surface, pure reflection was used and at least for the case of wedge flow proves to be satisfactory as it is an exact boundary condition.

This flow is a conical flow and a solution is obtained by integrating in the radial direction starting from some initial data surface until changes in the flow variables along radial lines for each integration step become smaller than some acceptable limit. Pressure was used as a test variable and the difference in pressure for ten consecutive integration steps was required to be less than 0.1 % at each point in the computational mesh. If this criterion was satisfied, the solution was considered to have converged.

The results provided by solving Burgers' equation have shown that acceptable solutions can be obtained with either second- or third-order methods if the Courant number is near one. For this reason results for a Courant number of 0.3 are presented since the off design performance of each method is of major interest here.

Figure 7 shows the pressure distribution obtained for a wedge half angle of 7.5° at a free stream Mach number of 2 using both MacCormack's and the K–L–W method. For MacCormack's method, a particularly large number of oscillations occur on both sides of the shock at this low Courant number. The shock wave is positioned very well in comparison to the exact location, and the shock layer pressures are correct. The large oscillations near the shock do cause significant deviations from the exact solution and lead to a large error estimate if a measure of the quality of the solution is taken as

$$\text{Error} = \sum_j | P(j) - P_{\text{exact}} |. \tag{20}$$

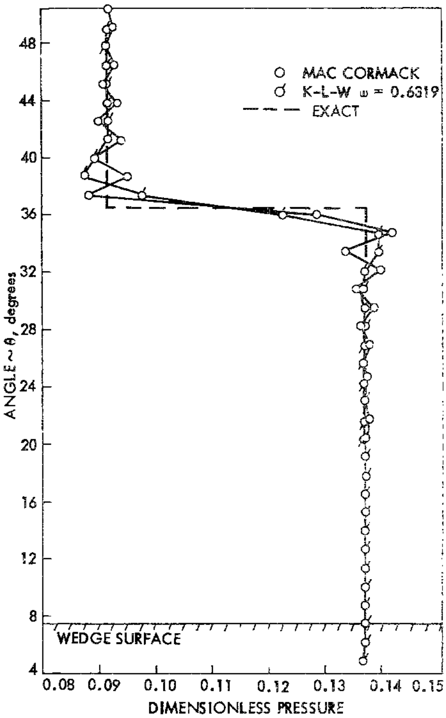

FIG. 7. Wedge flow solution $\Delta\xi/\Delta\theta = 0.3$, $\nu = 0.3$, 100 iterations.

That is, the difference between the calculated and exact solutions is determined at each mesh point and this difference is summed over all points in the field. The total error for this particular case was 0.0904.

Using the K–L–W method for a Courant number of 0.3 with the ω parameter held constant and equal to its value at the lower stability bound, the shock is spread over approximately three mesh intervals and is not as sharply defined as that produced by the MacCormack method. However, the absence of oscillations, both pre- and post-shock, makes this a better solution. The deviation from the exact solution as predicted by Eq. (20) is 0.0326 which is significantly lower than the previous case.

Figure 8 presents the solution using the tuned K–L–W method for minimum dispersion and dissipation respectively. The results for both cases are similar with small differences in peak over shoot and shock sharpness appearing. The deviation from the exact solution of the minimum dissipation case was calculated to be 0.0307 while that for minimum dispersion was 0.0443. As in the solution to Burgers' equation, the K–L–W method tuned for minimum dissipation provides the best results. However, the fixed ω case is nearer the exact solution than that obtained

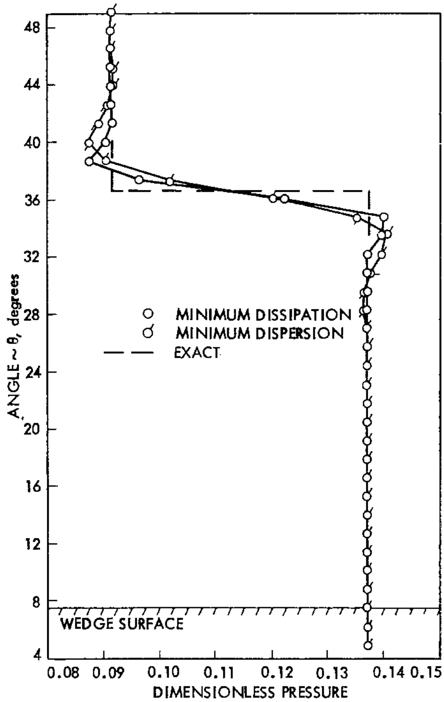

FIG. 8. Wedge flow solution using tuned K–L–W method $\Delta\xi/\Delta\theta = .3$, $\nu = .3$.

using minimum dispersion. Since the differences among the third-order solutions are small, satisfactory results should be obtained using any of the three.

SHOCK REFLECTION FROM A SOLID BOUNDARY

An additional example denoting the importance of the variation of local Courant number is provided by reflection of an oblique shock wave from a solid boundary. This is a well defined problem in which the solution for the reflected shock wave angle and the downstream flow is uniquely determined by the downstream boundary condition. The solution is termed a regular reflection if the reflected shock is within the attached shock region for two-dimensional flow, and it is termed Mach reflection if the reflected shock angle is required by the downstream boundary conditions to exceed the maximum allowable angle for an attached shock. The Mach reflection case is not of interest since the steady flow equations become elliptic in a portion of the downstream flow field. The regular reflection case retains the hyperbolic character of the equations throughout.

The procedure used in obtaining a solution to the shock reflection problem is to initialize conditions along an x = constant line. The x-coordinate is assumed positive in the direction of the original free stream which is parallel to the wall. Initial data input include both pre- and post-chock values of the dependent variables in the flow field while the usual reflection boundary condition is used at the solid boundary.

With rectangular coordinates along and normal to the original free stream, the maximum step size in the integration process is determined by the solution downstream of the incident shockwave. Therefore, a set of initial conditions producing a shock close to the detachment region is important in producing a large variation in eigenvalue structure. This problem produces a more severe test of a differencing method than either the wedge or the solution of Burgers' equation. The method must work well for a wide variation of mesh ratios and in addition must operate near the boundary where the character of the describing Eqs. (19) changes from hyperbolic to elliptic. It should be noted, however, that the influence of the mesh ratio size and the switch from hyperbolic to elliptic are closely related since both the maximum mesh ratio and the character of the equations are determined from the eigenvalues.

Figures 9–13 present results obtained for an original free stream Mach number of 4 and an initial shock wave angle of 34° with respect to the free stream. The maximum allowable mesh ratio in each region is

$$\left(\frac{\Delta x}{\Delta y}\right)_{\max} = \frac{u^2 - c^2}{|uv| + c(u^2 + v^2 - c^2)^{1/2}},$$

where c is the speed of sound.

$$\text{Region I} \quad \left(\frac{\Delta x}{\Delta y}\right)_{\max} = 3.96843,$$

$$\text{II} \quad \left(\frac{\Delta x}{\Delta y}\right)_{\max} = 0.993475,$$

$$\text{III} \quad \left(\frac{\Delta x}{\Delta y}\right)_{\max} = 1.19779.$$

The mesh ratio used was 0.20, approximately 20% of the maximum allowable for stability in Region II.

It is of interest to alter the Rusanov method and make it either minimum dispersive or minimum dissipative by conservatively differencing the omega term in the third level. The third level of the K–L–W and Rusanov cases are then identical for cases where the omega parameter is either fixed or variable.

Figure 9 shows results using MacCormack's method while Figs. 10 and 11 present calculations using the K–L–W method with the stability parameter at its extremes.

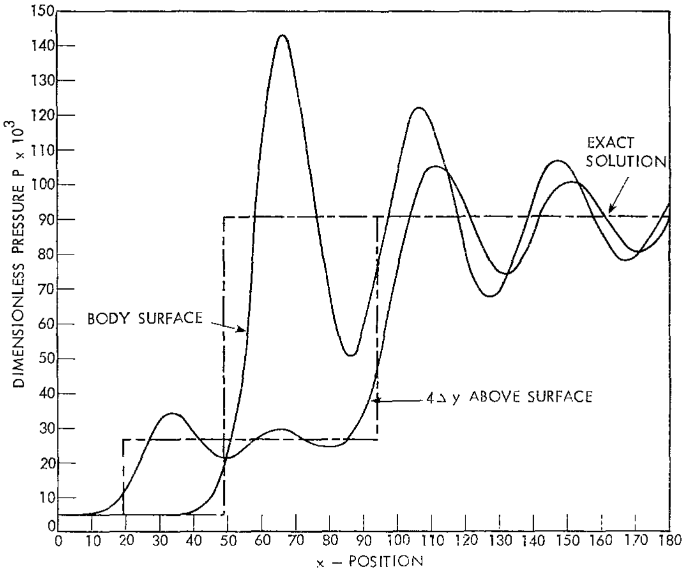

FIG. 9. Shock reflection problem, MacCormack method $\Delta x/\Delta y = 0.2$, $\theta_s = 34°$, $M_\infty = 4.0$.

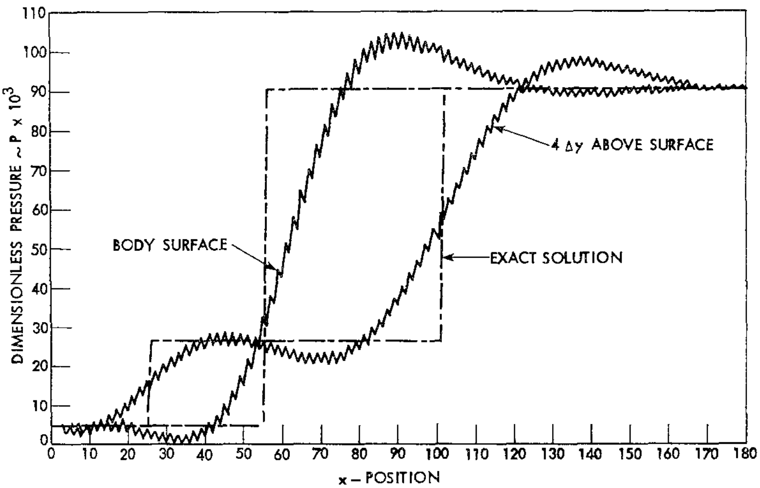

FIG. 10. Shock reflection problem, K–L–W method. $\omega = 3.0$, $\Delta x/\Delta y = 0.2$, $\theta_s = 34°$.

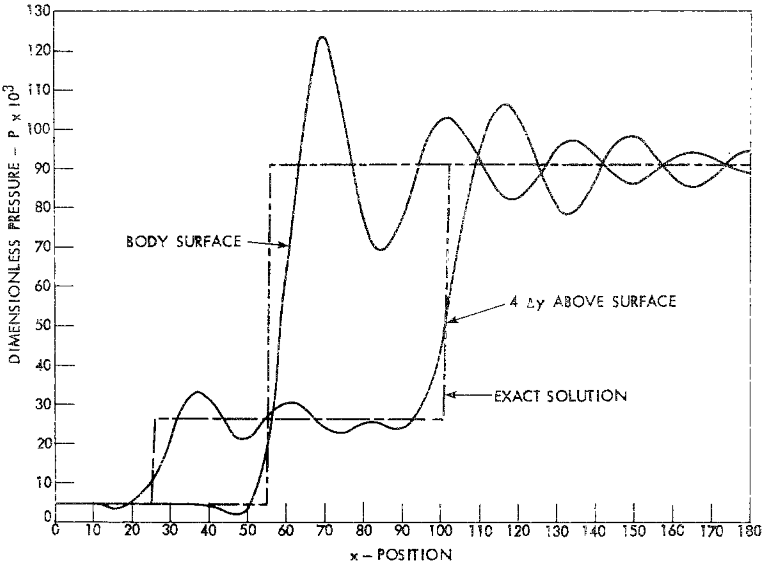

FIG. 11. Shock reflection problem, K–L–W method. $\omega = 0.1584$, $\Delta x/\Delta y = 0.20$, $\theta_s = 34°$.

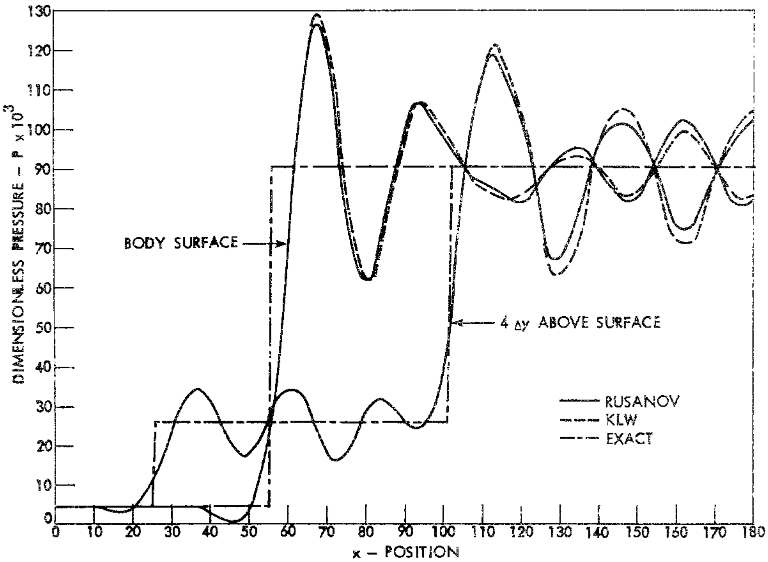

FIG. 12. Shock reflection problem, minimum dissipation. $\Delta x/\Delta y = 0.2$, $\theta_2 = 34°$, $M_\infty = 4.0$.

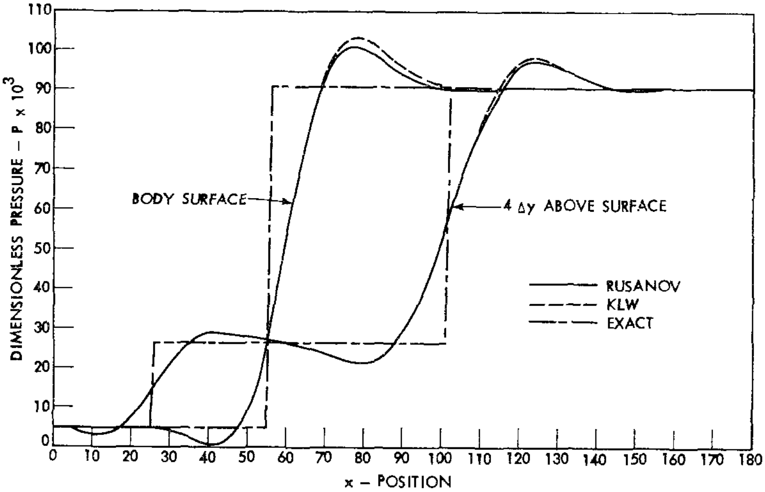

FIG. 13. Shock reflection problem, minimum dispersion. $\Delta x/\Delta y = 0.20$, $\theta_s = 34°$, $M_\infty = 4.0$.

Results obtained using Rusanov's method with fixed omega are not included since they are essentially identical for this problem. The K–L–W solution with $\omega = 3.0$ clearly presents a closer approximation of the exact solution if average values of pressure are used where the high frequency oscillations occur. This behavior is common to the K–L–W and Rusanov methods when the Courant number is relatively low and the upper limit value of ω is used. Figure 12 presents results using both the Rusanov and K–L–W methods tuned for minimum dissipation while Fig. 13 shows results for minimum dispersion. The best solution obtained is clearly the minimum dispersion case. It has the least overshoot of any of the techniques evaluated and still properly positions the shock waves. It should be noted that no estimate of the error was made for this problem. If an incident shock wave angle higher than 34° is used, the minimum dissipation calculations immediately become unstable. This is due to the absence of sufficient numerical damping coupled with operation near the elliptic boundary of the system. The minimum dispersion and dissipation solutions for this case show that Rusanov's method actually produces a better result than the K–L–W method. The difference is small but the Rusanov method produces less overshoot when it is tuned.

CONCLUSIONS

The comparison of results obtained using MacCormack's and either the K–L–W or Rusanov methods shows that second-order methods give acceptable results for most cases. When operating near a Courant number of one, the second-order

MacCormack method provides the best results from among those tested for the examples investigated in this paper. Third-order techniques provide the best results for the same example problems when the Courant number varies appreciably in the computational mesh.

The K–L–W method with minimum dissipation provided excellent results for the inviscid Burgers' equation and wedge flows but was not a satisfactory method for the shock reflection problem. The absence of numerical damping causes unacceptable behavior when rapid changes in the dependent variable are encountered such as shock waves or other discontinuities. Of the third-order methods tested, the minimum dispersion K–L–W or Rusanov method provides the most accurate results over the widest range of effective Courant numbers. This is consistent with conclusions presented in [6]. However, problems involving convection which do not have discontinuities or other rapid changes in the dependent variables appear to support a different conclusion [1].

It is recommended that the tuned minimum dispersion version of either the K–L–W or Rusanov technique be used for computations in problems which have large variations in the eigenvalue structure. The K–L–W method is recommended over the Rusanov method due to simplicity in programming even though slightly better results are obtained using the latter technique.

It should be noted that the results presented here are compared on the basis of the same number of grid points in the computational field. If the techniques are compared on the basis of equal computational times, then better spatial resolution can be obtained with lower order methods. The overall solution accuracy should be comparable for both second- and third-order methods in that event.

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